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# Analytical form of the Probability Density Function of travel times in rectangular zone with Tchebyshev's metric

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**Geometrical dependencies are being researched for analytical representation of the probability density function (pdf) for the travel time between a random, and a known or another random point in Tchebyshev's metric. In the most popular case – a rectangular area of service – the pdf of this random variable depends directly on the position of the server. Two approaches have been introduced for the exact analytical calculation of the pdf: Ad-hoc approach – useful for a 'manual' solving of a specific case; by superposition – an algorithmic approach for the general case. The main concept of each approach is explained, and a short comparison is done to prove the faithfulness.**

*[Keywords: isochrones, pdf, random trip, travel time, Tchebyshev metrics]*

## 1 Introduction

The automated storage and retrieval systems are commonly used world-wide. Usually they consist of racks served by cranes running through aisles between the racks. The cranes are capable of moving both vertically and horizontally simultaneously, thus their travel time is following the rules of the Tchebyshev's metric, in which the travel time of the path between two points 1 and 2 (in 2D area) is equal to the maximum of $t_{|x1-x2|}$ and $t_{|y1-y2|}$. Another example of Tchebyshev's metric is container storage and retrieval terminals, where the crane runs over a block of containers, and its wagon moves simultaneously.

The first research on the travel times in this metric was done in the early 1970's by Gudehus [Gud72], [Gud73]. However, in his survey from 2009 Roodbergen points out Hausman, Schwarz and Graves [Hsg76] as the first authors researching the travel time in 1976. The same authors extended their first simple travel time model, and suggest estimations for storage-retrieval systems with interleaving [Ghs77]. Later the same model was considered by Bozer and White with relaxations of the assumptions [Boz84]. Travel time is estimated for rectangular racks, and the rack geometry is represented by a single variable (shape factor) instead of two (width and height). Bozer and White suggest a set of equations for travel time estimation, based on the shape factor.

As Roodbergen [Roo09] points out, other authors since then mainly continue the research of Hausman et al. (1976), Graves et al. (1977) and Bozer and White (1984) by studying different control policies, configurations of AS/RSs and/or operational characteristics. These works are used as a foundation for many scientific articles around the world – Russia (Smehov 1997 [Sme97]), South Korea (Park 1992 [Par92]), USA (Park 2002 [Par02]) and others.

All those materials examine different aspects of the problems in Tchebyshev's metric, as for example single and double cycles, mean travel time and others. The articles explain dependencies of the main probability characteristics, including the mean and variance. The probability density function for an I/O point located at the (lower-left hand) corner is known from [Boz84]. In the same paper the authors consider alternative configurations for the input and output points: At the same end, but at different elevations; at opposite ends; or at the midpoint at the same end of the isle.

## 2 Motivation

In general the server may start traveling from any location, either known (as dwell point or last storage/retrieval location) or unknown (future storage/retrieval or random location). Such a trip may be followed by another one, thus forming a continuous traveling from one location to another one. This is an ordinary person-on-board picking scenario, where the variable of interest is travel time between picking locations. Such a scenario matches the M|G|1 queuing model, where the requests arrive exponentially in time, but are served on a general distribution.

The mean and variance of the travel time for such picking trips are expressed for a dual command cycle in [Par91]. However, an exact examination of M|G|1 queues requires that the general distribution to be expressed in some manner. In the best case, the probability density function of the travel time will be defined. With pdf known, the mean, the variance and moments of higher order could be calculated.

A recent paper [Tod06] reexamines rectangular service area with end-of-aisle picking, for an I/O point located at the lower-left hand corner. The article presents an alternate approach for calculation of the probability density function of the travel time, and the results comply with those of [Boz84].

The calculation suggested by [Tod06] considers the area of isochrone lines as geometrical probability. The present article evolves this approach for expressing probability density function of travel time, if the I/O point is located at any position.

## 3 RANDOM AND SEMI-RANDOM TRIP

For clarity and accuracy, in the present paper the terms below are used exclusively:

- *Single trip* is a travel from a start point to end point, through the shortest possible distance.
- *Random trip* is a single trip from a random point to a random point.
- *Semi-random trip* is a single trip which starts or ends with a random point, but the opposite side of the trip is a fixed point (yet arbitrary) which coordinates are known.

Using these terms, the dual command cycle could be considered as a sequence of three single trips: semi-random (starting from Input location), random, semi-random (ending at Output location).

## 4 I/O POINT LOCATED AT THE CORNER

The problem of I/O point located at one of the corners of a rectangular service zone (hereinafter referred to as "corner case") is examined in detail in [Tod06] for normalized time domain with limits *1* and *shape factor b*. As the suggested approach does not require normalization, the approach is presented below for non-normalized time domain.

A rectangular area of service is given (fig.1) in non-normalized time domain with size $T_x$ and $T_y$. When the I/O point is located at the lower-left hand corner (position *0*), the isochrones[1] are either L-shaped or I-shaped thin surfaces, respectively in the intervals $t<T_{MIN}$ and $T_{MIN}<t<T_{MAX}$ (where $T_{MAX}=max(T_x,T_y)$ and $T_{MIN}=min(T_x,T_y)$). The area of an isochrone surface on distance *t* from the I/O point divided by the area of the whole service zone represents the probability that a single travel with duration between *t* and *t+dt* will occur, where *dt* is the width of the isochrone.

[1] Isochrone - a line connecting places to which it takes the same time to travel. In the present paper all isochrones are calculated according to the I/O point.

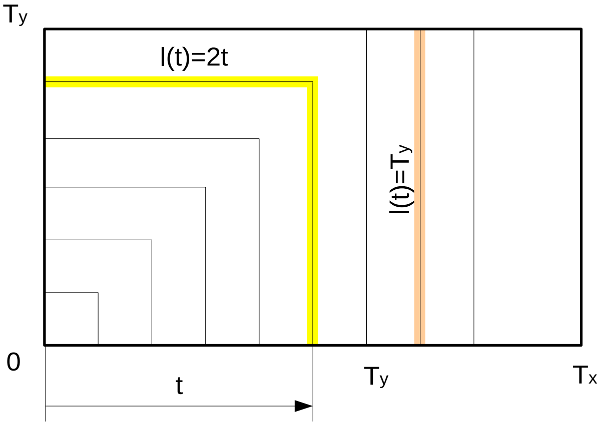


*Figure 1. Service area in a non-normalized time domain*

Figure 2 shows the density of the time for a semi-random travel from point *0* to a random point in the zone, which is the random variable[2] $t=max(\tau_x,\tau_y)$. The probability density function[3] $f_K(t)$ of this variable is initially shown in [Tod06] as the relation between the area $ds(t)$ of an isochrone and the service area $S=T_x.T_y$. Taking into account that $f_K(t)=dF_K(t)/dt$, this relation could be further simplified to a relation of the isochrones' lengths to the service area (still in time domain).

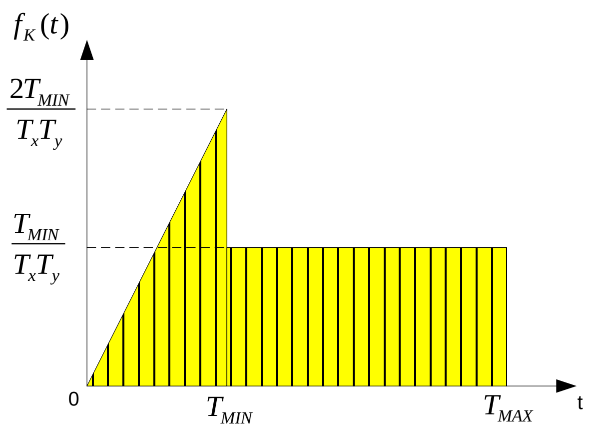


*Figure 2. Pdf of the travel time for semi-random trip*

The transformation of equation (1) is done with the following observation: the surface $ds(t)$ of each isochrone in Tchebyshev's metric is equal to the product of the isochrone's length and its width (equal to the increase *dt* of the argument): $ds(t)=l(t)dt$.

[2] The travel back from the same random point to point *0* is a semi-random travel with the same characteristics. Hence, the travel time for single-command cycle can be obtained by doubling the random variable *t*.

[3] The letter K in $f_K(t)$ is used only to remind that the trip starts from (or ends to) a point with *Known* coordinates.

$$l(t)=\begin{cases}2t & 0\le t\le T_{MIN}\\ T_{MIN} & T_{MIN}<t\le T_{MAX}\end{cases}\Leftrightarrow f_K(t)=\frac{ds(t)}{S.dt}=\frac{l(t).dt}{S.dt}=\frac{l(t)}{S}=\begin{cases}\dfrac{2t}{T_xT_y} & 0\le t\le T_{MIN}\\[2mm] \dfrac{T_{MIN}}{T_xT_y} & T_{MIN}<t\le T_{MAX}\end{cases} \quad (1)$$

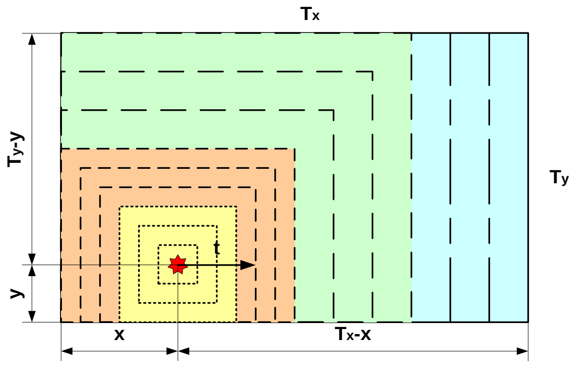


*Figure 3. Isochrones with different shape*

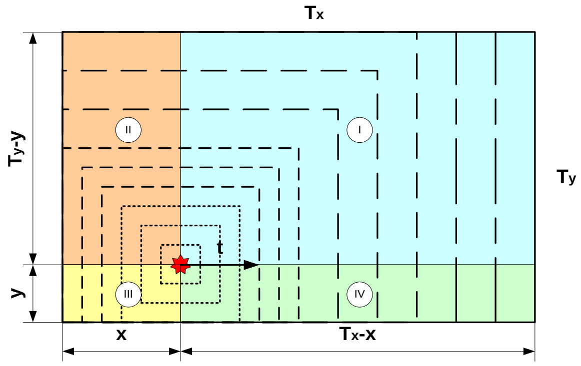


*Figure 4. Superposition of isochrones with same shape*

$$l(t)=l_I(t)+l_{II}(t)+l_{III}(t)+l_{IV}(t)\Leftrightarrow ds(t)=ds_I(t)+ds_{II}(t)+ds_{III}(t)+ds_{IV}(t) \quad (2)$$

$$l_I(t)=\begin{cases}2t & 0\le t\le T_y-y\\ T_y-y & T_y-y<t\le T_x-x\end{cases}\quad l_{II}(t)=\begin{cases}2t & 0\le t\le x\\ x & x<t\le T_y-y\end{cases}$$
$$l_{III}(t)=\begin{cases}2t & 0\le t\le y\\ y & y<t\le x\end{cases}\quad l_{IV}(t)=\begin{cases}2t & 0\le t\le y\\ y & y<t\le T_x-x\end{cases} \quad (3)$$

$$f_I(t)=\begin{cases}\dfrac{2t}{T_xT_y} & 0\le t\le T_y-y\\[2mm] \dfrac{T_y-y}{T_xT_y} & T_y-y<t\le T_x-x\end{cases}\quad f_{II}(t)=\begin{cases}\dfrac{2t}{T_xT_y} & 0\le t\le x\\[2mm] \dfrac{x}{T_xT_y} & x<t\le T_y-y\end{cases}$$
$$f_{III}(t)=\begin{cases}\dfrac{2t}{T_xT_y} & 0\le t\le y\\[2mm] \dfrac{y}{T_xT_y} & y<t\le x\end{cases}\quad f_{IV}(t)=\begin{cases}\dfrac{2t}{T_xT_y} & 0\le t\le y\\[2mm] \dfrac{y}{T_xT_y} & y<t\le T_x-x\end{cases} \quad (4.1)$$

## 5 I/O POINT AT ANY POSITION (SEMI-RANDOM TRIP)

If the I/O point is located at an arbitrary position *(x,y)* as shown in Figure 3, the isochrones could have up to 4 different shapes (□, ∏, L, or I-shape), for the different possible intervals of the distance *t*. In every interval the length *l(t)* of an isochrone can be expressed by *t*, $T_x$ and $T_y$. Then from the dependency $f_K(t)=l(t)/S$ the probability density function of the travel time can be obtained, as it was done in equation (1) for an I/O point located at the corner. This approach is an exact application of the method described in [Tod06], as in this case the lengths of the isochrones are used, instead of their surfaces.

This approach has the following disadvantage: The form of the isochrones in the different intervals depends on the position of the server, leading to a dependence of the expressions on logical decisions. However, there is an alternative approach. The I/O located anywhere can be considered as a superposition of four "corner cases", as it is shown on Figure 4.

Wherever it is located, the I/O point can be viewed as a common (border) point between four rectangular sub-areas. For every sub-area the I/O point is always located at a corner. The length of the complete isochrone is considered as a sum of all its partial lengths (up to four) with each part residing in a different subarea. The same is valid for the surfaces of the isochrones - equation (2).

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As in ordinary "corner case", every part of an isochrone can be either L- or I-shaped in a given subarea. Furthermore, equation (1) remains always valid, taking into account the borders of the subarea. For the special case[4] shown on figure 4 the lengths of the isochrones are calculated in equation (3) and the pdf *f(t)* of the travel time is shown in equation (4.1). What remains to be calculated is the sum for all intervals with equation (4.2):

$$f(t) = \sum_{i=1}^{4} f_i(t) \qquad (4.2)$$

The superposition of different sub-areas allows further simplifying of the calculation, as follows:

Equation (1) gives the dependency of the pdf (or the isochrone lengths/surfaces) on the limits of the argument *t*, and on the size of the service area. For the ordinary corner case the values $T_{MIN}$ and $T_{MAX}$ match the size of the area. These values can also be assumed to be points of change of the isochrones' shape. With such assumptions equation (1) describes each of the four subareas, and the equation can be parameterized as $f_i(t) = f_i(t, T_{MIN}, T_{MAX})$.

Then the probability density function of the travel time from an I/O located at point K with known coordinates *(x,y)* is calculated in the general case[5] as in equation (5):

$$\begin{aligned} f_K(t,x,y) = {} & f_I(t, \min(T_x - x, T_y - y), \max(T_x - x, T_y - y)) \\ & + f_{II}(t, \min(x, T_y - y), \max(x, T_y - y)) \\ & + f_{III}(t, \min(x,y), \max(x,y)) \\ & + f_{IV}(t, \min(T_x - x, y), \max(T_x - x, y)) \end{aligned} \qquad (5)$$

## 6 GRAPHICAL REPRESENTATION OF THE PDF

The graphic of the probability density function of the time for a single travel from an I/O located in the corner is known since the seventies. However, until now what the pdf looks like when an I/O point is located at an arbitrary (known) point *K* has not been published. The graphical representation of this function is shown in figure 5 and figure 6, together with the graphic's behavior in the different subareas. The $f_K(t)$ has been calculated for a rectangular service area with size $T_x=100$, $T_y=60$; and an I/O point located at coordinates $x=25$, $y=15$. Tchebyshev's metric is considered.

The result function $f_K(t)$ in figure 5 is calculated according to the division shown in figure 3, for which every isochrone resides completely in the same sub-area. The result graphic is a piece-wise function, composed by 4 different functions, each of which represents the *l(t)/S* relation of the corresponding subinterval of the varying of *t*. These intervals are: (0,15]; (15;25]; (25, 60-15] and (60-15, 100-25].

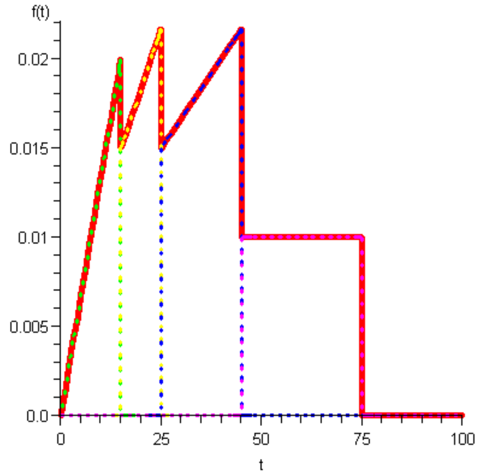


*Figure 5. Pdf calculated as piecewise function through isochrones with different shapes*

In contrast, the graphic $f_K(t)$ in figure 6 is composed as a superposition of $f_i(t)$ functions defined for the four subareas as shown in figure 4. In every sub-area t varies from 0 to $T_{MAX}$ the rule of equation (1).

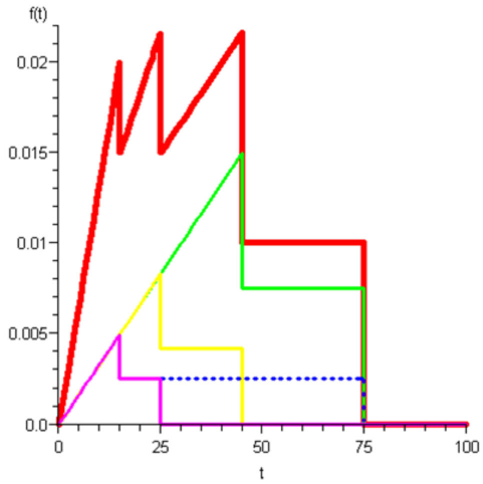


*Figure 6. Pdf calculated as superposition of isochrones with same shape*

The graphic $f_K(t)$ is the same in both figures – a three-peak left part, plus an uniform right part. From figure 6 it is clear why the shape looks like that. It is because in the four sub-areas, the pdf has the same appearance (as in figure 2). The peaks of the two subareas always coincide, as the shortest distance $T_{MIN}$ between the I/O location and the area borders is always common for two subareas (in the given example – these are subareas III and IV).

[4] Equations (3) and (4.1) are valid when the values of *x, y,* $T_x$*-x* and $T_y$*-y* relate to each other as shown on figure 4 (*y*<*x* etc.)

[5] Not restricted to the special case on figure 4

## 7 COMPARISON WITH EXISTING METHODS (RANDOM TRIP)

The density function defined with equation (5) represents the travel time of a semi-random trip in the service area. In order to evaluate the correctness of the equation, the mean of a random travel will be calculated and compared with the result derived through an existing method. Based on this pdf, one can write integrals (6.1) and (6.2).

$$E_{SRT}(x,y) = \int_0^{T_{MAX}} tf(t,x,y)dt \tag{6.1}$$

$$E_{SRT2}(x,y) = \int_0^{T_{MAX}} t^2 f(t,x,y)dt \tag{6.2}$$

Expressions (6.1) and (6.2) give the mean of travel time and square travel time for a semi-random trip from *(x,y)* to a random location. As *(x,y)* is an arbitrary location within the service area, these expressions define all semi-random trips for this area.

$$E_{RT} = \frac{1}{T_X T_Y} \iint_{T_X T_Y} E_{SRT}(x,y)dxdy \tag{6.3}$$

$$E_{RT2} = \frac{1}{T_X T_Y} \iint_{T_X T_Y} E_{SRT2}(x,y)dxdy \tag{6.4}$$

$$Var_{RT} = E_{RT2} - (E_{RT})^2 \tag{6.5}$$

Expressions (6.3) and (6.4) apply the semi-random calculations by placing the I/O point at every possible location within the service area, then averaging the integral sum. As a result the means of travel time and square travel time for a *random trip* (from random to random location) within the area are obtained. Finally, equation (6.5) gives the variance of the travel time for a random trip by using (6.3) and (6.4).

These values can be compared to already known results published in [Par91] as shown in Table 1.

The above mentioned comparison is presented[6] in Table 2. Every third row of the table presents the absolute values of the calculated percentage deviations of the corresponding values. One can see that the maximal deviation between the results of Park and Todorov is less than 0.03%. Such deviation can be attributed to the computing error accumulated during the triple numeric integration of *f(t,x,y)*. Taking this into account, the results match.

From the coincidence of the results it can be concluded that the aforementioned equations are correct. Especially interesting is equation (5) which allows the probability density function of the travel time to be calculated in a *non*-normalized time domain. The sample graphic shown on Figure 5 and Figure 6 has not been published until now. All subsequent equations are also defined in a non-normalized time domain. They do not depend on the shape factor *b* suggested by [Boz84], and the results obtained (e.g. by (6.1)) will be in the exact time units.

## 8 CONCLUSION

The present research contributes in the following: It improves the method presented in [Tod06] by defining the probability density function of travel time without normalization. The research also introduces two approaches for the exact analytical calculation of the probability density function with I/O points located at any position. The comparison made between these approaches suggests that the ad-hoc approach is useful for a ‘manual’ solving, but the superposition approach is applicable for an automated solving of the general case. Further comparison is carried out between the results obtained by the method suggested by Todorov and those obtained by Park. According to the comparison the methods are equivalent, which proves the faithfulness of the suggested approach.

[6] The shape factor *b* for $E_T$ is calculated as $b=T_{MIN}/T_{MAX}$, with $T_{MAX}$=10 and $T_{MIN}$ varying from 1 to 10. The calculation is done by Maple, and the precision is limited to 4 decimal digits because of the high-computing demand of the triple integral.

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*Table 1. Values to compare*

| **Meaning of the value** | **Notation** | **Equation** | **Notation in [Par91]** | **Equation in [Par91]** |
|---|---|---|---|---|
| Mean of travel time for a random trip | $E_{RT}$ | (6.3) | E[D] | (2.20) |
| Mean of square travel time for a random trip | $E_{RT2}$ | (6.4) | $E[D^2]$ | (2.23) |
| Variance of travel time for a random trip | $Var_{RT}$ | (6.5) | $E[D^2]-E[D]^2$ | *n/a* |

*Table 2. Comparison of the results*

| ***b*** | **0.1** | **0.2** | **0.3** | **0.4** | **0.5** | **0.6** | **0.7** | **0.8** | **0.9** | **1.0** |
|---|---|---|---|---|---|---|---|---|---|---|
| ***E[D]*** | 0.3349 | 0.3397 | 0.3474 | 0.3578 | 0.3708 | 0.3861 | 0.4035 | 0.4229 | 0.4440 | 0.4667 |
| $E_{RT}$ | 0.3350 | 0.3398 | 0.3473 | 0.3578 | 0.3708 | 0.3862 | 0.4036 | 0.4229 | 0.4440 | 0.4667 |
| $\lvert\Delta\%\rvert$ | 0.0299 | 0.0294 | 0.0288 | 0 | 0 | 0.0259 | 0.0248 | 0 | 0 | 0.0294 |
| $E[D^2]$ | 0.1668 | 0.1677 | 0.1700 | 0.1744 | 0.1813 | 0.1912 | 0.2044 | 0.2213 | 0.2420 | 0.2667 |
| $E_{RT2}$ | 0.1668 | 0.1677 | 0.1700 | 0.1744 | 0.1812 | 0.1912 | 0.2044 | 0.2212 | 0.2420 | 0.2667 |
| $\lvert\Delta\%\rvert$ | 0 | 0 | 0 | 0 | 0.0552 | 0 | 0 | 0.0452 | 0 | 0 |
| $E[D^2]-E[D]^2$ | 0.0546 | 0.0523 | 0.0493 | 0.0464 | 0.0438 | 0.0421 | 0.0416 | 0.0425 | 0.0449 | 0.0489 |
| $Var_{RT}$ | 0.0546 | 0.0522 | 0.0494 | 0.0464 | 0.0437 | 0.0420 | 0.0415 | 0.0424 | 0.0449 | 0.0489 |
| $\lvert\Delta\%\rvert$ | 0.1226 | 0.1299 | 0.1409 | 0 | 0.2282 | 0.1833 | 0.1941 | 0.2355 | 0 | 0 |

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